Kinetic models of opinion formation in the presence of personal conviction

Carlo Brugna* and Giuseppe Toscani†

Dipartimento di Matematica, Università di Pavia, Via Ferrata 1, 27100 Pavia, Italy.

(Dated: March 26, 2015)

We consider a nonlinear kinetic equation of Boltzmann type which takes into account the influence of conviction during the formation of opinion in a system of agents which interact through the binary exchanges introduced in [G. Toscani, *Commun. Math. Sci.* **4**, 481(2006)]. The original exchange mechanism, which is based on the human tendency to compromise and change of opinion through self-thinking, is here modified in the parameters of the compromise and diffusion terms, which now are assumed to depend on the personal degree of conviction. The numerical simulations show that the presence of conviction has the potential to break symmetry, and to produce clusters of opinions. The model is partially inspired by the recent work [L. Pareschi, G. Toscani, *Phil. Trans. R. Soc. A* **372**, 20130396 (2014)], in which the role of knowledge in the formation of wealth distribution has been investigated.

PACS numbers: 89.65.Gh, 87.23.Ge, 05.10.-a

I. INTRODUCTION

In recent years, the dynamics of opinion formation in a multi-agent society has received growing attention [1–7]. In reason of its cooperative nature, it appeared natural to resort to tools and methods typical of statistical mechanics to study such systems [8, 9]. The approaches considered so far range from cellular automata, especially used for numerical simulation, to models of mean field type, which lead to systems of (ordinary or partial) differential equations, to kinetic models of opinion formation [10–15]. In kinetic models, the variation of opinion is obtained through binary interactions between agents. In view of the relation between parameters in the microscopic binary rules, the society develops a certain steady macroscopic opinion distribution [16, 17], which characterizes the formation of a relative consensus around certain opinions.

The relevant aspects to be taken into account when modelling binary interactions in opinion formation have been identified in the compromise process [4, 18], in which individuals tend to reach a compromise after exchange of opinions, and the self-thinking [15], where individuals change their opinion in a diffusive fashion.

Following this line of thought, a wide class of kinetic models of opinion formation, based on two-body interactions involving both compromise and diffusion properties in exchanges between individuals, has been introduced in [15]. These models are sufficiently general to take into account a large variety of human behaviors, and to reproduce in many cases explicit steady profiles from which one can easily elaborate information on the macroscopic opinion distribution. This type of modelling has subsequently been applied to various situations in [11–13, 19].

More recently, a further relevant parameter, strongly related to the problem of opinion formation, has been taken into account in [20]. Resembling the model for wealth exchange in a multi-agent society introduced in [21], this new has an additional parameter to quantify the personal *conviction*, representing a measure of the influencing ability of individuals. Individuals with high conviction are resistant to change opinion, and have a prominent role in attracting other individuals towards their opinions. In this sense, individuals with high conviction play the role of leaders [19].

The role of conviction has been subsequently considered by other authors. It was shown in [20] that beyond a certain value of this conviction parameter, the society reaches a consensus, where one of the two choices (positive or negative) provided to the individuals prevails, thereby spontaneously breaking a discrete symmetry. A further model in which this parameter has been taken into account was proposed in [22]. There, the self-conviction and the ability to influence others were taken as independent variables. Also, exact solutions of a discrete opinion formation model with conviction were found by Biswas in [23]. In [24, 25], conviction has been introduced as relevant parameter in a class of discrete opinion models. Within this class, each agent opinion takes only discrete values, and its time evolution is ruled by two terms, one representing binary interactions between individuals, and the other the degree of conviction or persuasion (a self-interaction).

In all the aforementioned models, conviction is realized by a fixed-in-time parameter (or a random variable), eventually different for different individuals. Consequently, while it is clear that a certain distribution of this parameter among agents leads to a steady distribution of opinions with properties which are related to it, it is not completely clarified why conviction has to be assumed with a certain distribution.

All these studies, however, indicate that, among the

*Electronic address: cabrugn@tim.it
†URL: www-dimat.unipv.it/toscani; Electronic address: giuseppe.toscani@unipv.it

various behavioral aspects that determine a certain opinion formation, conviction represents an important variable to be taken into account. This problem has deep analogies with a recent study of one of the present authors with Pareschi [26], where the role of knowledge in wealth distribution forming has been studied by allowing agents to depend on two variables, denoting knowledge and wealth respectively. There, a kinetic equation for the evolution of knowledge has been coupled with a kinetic model for wealth distribution, by allowing binary trades to be dependent of the personal degree of knowledge. *Mutatis mutandis*, we will assume in this paper that individuals are characterized by two variables, representing conviction and, respectively, opinion. Following the line of thought in [26], we will introduce here a kinetic model for conviction formation, by assuming that the way in which conviction is formed is independent of the personal opinion. Then, the (personal) conviction parameter will enter into the microscopic binary interactions for opinion formation considered in [15], to modify them in the compromise and self-thinking terms. Within this picture, both the conviction and the opinion are modified in time in terms of microscopic interactions. As we shall see by numerical investigation, the role of the additional conviction variable is to bring the system towards a steady distribution in which there is formation of clusters even in absence of bounded confidence hypotheses [1–3, 27].

In our kinetic model for conviction formation, we will assume that the relevant terms responsible of the modification of the conviction are from one side the the acquisition of information, and from the other side the possibility of afterthought and to think back, which appear to be natural and universal features. In reason of this, the positive parameter that quantifies conviction can either increase (through information) or decrease (through afterthought). The relevant aspect is that information will be achieved though the surrounding, thus producing a linear kinetic model. Note that it is assumed here that the individual conviction is not correlated to personal opinion, thus allowing formation of conviction without resorting to the distribution of opinions. Then, the linear conviction interaction will be coupled with the binary exchange of opinion introduced in [15], which includes both the compromise propensity and the the change of the personal opinion due to self-thinking. In the new interaction rule both the compromise and the self-thinking part of the opinion exchange depend on the personal conviction. A typical and natural assumption is that high conviction could act on the interaction process both to reduce the personal propensity to compromise, and to reduce the self-thinking in the interaction. These rules will subsequently be merged, within the principles of classical kinetic theory, to derive a nonlinear Boltzmann-like kinetic equation for the joint evolution of conviction and opinion variables.

The kinetic approach revealed to be a powerful tool [17, 28], complementary to the numerous theoretical and numerical studies that can be found in the recent physical and economic literature on these subjects. On the other hand, as many other approaches, the study of the socio-economic behavior of a (real) population of agents by means of kinetic models with very few (essential) parameters is able to capture only partially the extremely complex behavior of such systems. The idea to introduce as additional parameter the conviction in the study of opinion formation goes exactly into the direction to give a more accurate description of the human behavior. Not surprisingly, the description of the evolution of pair conviction–opinion in terms of a kinetic equation gives rise to a variety of challenging mathematical problems, both from the theoretical and numerical point of view. In particular, concerning the distribution of conviction, it is remarkable that this class of simple models is able to reproduce various features always present in the reality, like the presence of a considerable number of undecided in the population, as well as the formation of clusters of opinions in the steady distribution.

To end this introduction, we outline that the problem of clustering is of paramount importance in this context. Indeed, models for opinion formation belong to the variety of models for self-organized dynamics in social, biological, and physical sciences [28], which assume that the intensity of alignment increases as agents get closer, reflecting a common tendency to align with those who think or act alike. As noticed by Motsch and Tadmor [16] *similarity breeds connection* reflects our intuition that increasing the intensity of alignment as the difference of positions decreases is more likely to lead to a consensus. However, it is argued in [16] that the converse is true: when the dynamics is driven by local interactions, it is more likely to approach a consensus when the interactions among agents increase as a function of their difference in position. In absence of further parameters, heterophily, the tendency to bond more with those who are different rather than with those who are similar, plays a decisive role in the process of clustering. This motivates further our choice to resort to the additional role of conviction.

The paper is organized as follows. In Section II we introduce and discuss the linear kinetic model for the formation of conviction in a multi-agent society. This linear model is based on microscopic interactions with a fixed background, and is such that the density of the population conviction converges towards a steady distribution which is heavily dependent of the microscopic parameters of the microscopic interactions. Then, the conviction rule is merged with the binary interaction for opinion to obtain a nonlinear kinetic model of Boltzmann-type for the joint density of conviction and opinion. This part is presented in Section III. Last, Section IV is devoted to various numerical experiments, which allow to recover the steady joint distribution of conviction and opinion in the population for various choices of the relevant parameters.

II. THE FORMATION OF CONVICTION

To give a precise and well-established definition of conviction is beyond our purposes. Instead of resorting to a definition it seems natural to agree on certain universal aspects about it. Conviction can be described as a certain resistance to modify a personal behavior. How the personal amount of conviction is formed is a very difficult question. We can reasonably argue that, among other reasons, responsible of conviction forming include familiar environment, personal contacts, readings or skills acquired through experience or education. It is natural to assume that conviction (at least concerning some aspects of life like religious or political beliefs) is in part inherited in the interior of family from the parents, but it is also evident that the the main factor that can influence it is the social background in which the individual grows and lives [29]. Indeed, the experiences that lead to be convinced about something can not be fully inherited from the parents, such as the eye color, but rather are acquired by several elements of the environment. This process is manifold and produces different results for each individual in a population. Like in knowledge formation, although all individuals are given the same opportunities, at the end of the process every individual appears to have a different level of conviction about something. Also, it is almost evident that the personal conviction is heavily dependent on the individual nature. A consistent part of us is accustomed to rethink, and to have continuous afterthoughts on many aspects of our daily decisions. This is particularly true nowadays, where the global access to information via web gives to each individual the possibility to have a *reservoir* of infinite capacity from which to extract any type of (useful or not) information, very often producing insecurity.

The previous remarks are at the basis of a suitable description of the evolution of the distribution of conviction in a population of agents by means of microscopic interactions with a fixed background. We will proceed as in [26]. Each variation of conviction is interpreted as an interaction where a fraction of the conviction of the individual is lost by virtue of afterthoughts and insecurities, while at the same time the individual can absorb a certain amount of conviction through the information achieved from the external background (the surrounding environment). In our approach, we quantify the conviction of the individual in terms of a scalar parameter x, ranging from zero to infinity. Denoting with $z \geq 0$ the degree of conviction achieved from the background, it is assumed that the new amount of conviction in a single interaction can be computed as

$$x^* = (1 - \lambda(x))x + \lambda_B(x)z + \kappa H(x). \tag{2.1}$$

In (2.1) the functions $\lambda = \lambda(x)$ and $\lambda_B = \lambda_B(x)$ quantify, respectively, the personal amounts of insecurity and willingness to be convinced by others, while κ is a random parameter which takes into account the possible unpredictable modifications of the conviction process. We will in general fix the mean value of κ equal to zero. Last, $H(\cdot)$ will denote an increasing function of conviction. The typical choice is to take $H(x) = x^\nu$, with $0 < \nu \leq 1$. Since some insecurity is always present, and at the same time it can not exceed a certain amount of the total conviction, it is assumed that $\lambda_- \leq \lambda(x) \leq \lambda_+$, where $\lambda_- > 0$, and $\lambda_+ < 1$. Likewise, we will assume an upper bound for the willingness to be convinced by the environment. Then, $0 \leq \lambda_B(x) \leq \bar{\lambda}$, where $\bar{\lambda} < 1$. Lastly, the random part is chosen to satisfy the lower bound $\kappa \geq -(1 - \lambda_+)$. By these assumptions, it is assured that the post-interaction value x^* of the conviction is nonnegative.

Let $C(z)$, $z \geq 0$ denote the probability distribution of degree of conviction of the (fixed) background. We will suppose that $C(z)$ has a bounded mean, so that

$$\int_{\mathbb{R}_+} C(z)\, dz = 1; \quad \int_{\mathbb{R}_+} z\, C(z)\, dz = M \tag{2.2}$$

We note that the distribution of the background will induce a certain policy of acquisition of conviction. This aspect has been discussed in [26], from which we extract the example that follow. Let us assume that the background is a random variable uniformly distributed on the interval $(0, a)$, where $a > 0$ is a fixed constant. If we choose for simplicity $\lambda(x) = \lambda_B(x) = \bar{\lambda}$, and the individual has a degree of conviction $x > a$, in absence of randomness the interaction will always produce a value $x^* \leq x$, namely a partial decrease of conviction. In this case, in fact, the process of insecurity in an individual with high conviction can not be restored by interaction with the environment.

The study of the time-evolution of the distribution of conviction produced by binary interactions of type (2.1) can be obtained by resorting to kinetic collision-like models [17]. Let $F = F(x, t)$ the density of agents which at time $t > 0$ are represented by their conviction $x \in \mathbb{R}_+$. Then, the time evolution of $F(x, t)$ obeys to a Boltzmann-like equation. This equation is usually written in weak form. It corresponds to say that the solution $F(x, t)$ satisfies, for all smooth functions $\varphi(x)$ (the observable quantities)

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}_+} F(x,t)\varphi(x)\, dx = \\ &\Big\langle \int_{\mathbb{R}_+^2} \big(\varphi(x^*) - \varphi(x)\big) F(x,t) C(z)\, dx\, dz \Big\rangle. \end{aligned} \tag{2.3}$$

In (2.3) the post-interaction conviction x^* is given by (2.1). As usual, $\langle \cdot \rangle$ represents mathematical expectation. Here expectation takes into account the presence of the random parameter κ in (2.1).

The meaning of the kinetic equation (2.3) is the following. At any positive time $t > 0$, the variation in time of the distribution of conviction (the left-hand side) results from a balance equation in which, through interaction with the background we gain agents with conviction x^*

loosing agents with conviction x. This change is measured by the interaction operator at the right-hand side.

In order to verify if the kinetic equation (2.3) gives reasonable outputs on the distribution of conviction among the population of agents, let us study some of its properties. It is immediate to recognize that equation (2.3) preserves the total mass, so that $F(x,t)$, $t>0$, remains a probability density if it is so initially. By choosing $\varphi(x)=x$ we recover the evolution of the mean conviction $M_C(t)$ of the agents system, which gives a first measure of the conviction rate. The mean value satisfies the equation

$$\frac{dM_C(t)}{dt} = -\int_{\mathbb{R}_+} x\lambda(x)F(t)\,dx + M\int_{\mathbb{R}_+} \lambda_B(x)F(t)\,dx, \tag{2.4}$$

which in general it is not explicitly solvable, unless the functions $\lambda(x)$ and $\lambda_B(x)$ are assumed to be constant. However, since $\lambda(x)\ge\lambda_-$, while $\lambda_B(x)\le\bar\lambda$, the mean value always satisfies the differential inequality

$$\frac{dM_C(t)}{dt} \le -\lambda_- M_C(t) + \bar\lambda M, \tag{2.5}$$

which guarantees that the mean conviction of the system will never exceed the (finite) value M_{max} given by

$$M_{max} = \frac{\bar\lambda}{\lambda_-}M.$$

If $\lambda(x)=\lambda$ and $\lambda_B(x)=\lambda_B$ are constant, equation (2.4) becomes

$$\frac{dM_C(t)}{dt} = -\lambda M_C(t) + \lambda_B M. \tag{2.6}$$

In this case, the linear differential equation can be solved, and

$$M_C(t) = M_C(0)e^{-\lambda t} + \frac{\lambda_B M}{\lambda}\left(1-e^{-\lambda t}\right). \tag{2.7}$$

Formula (2.7) shows that the mean conviction converges exponentially to its limit value $\lambda_B M/\lambda$. Note that by increasing the parameter λ which measures the personal amount of insecurity we decrease the final mean conviction.

A further insight into the linear kinetic equation (2.3) can be obtained by resorting to particular asymptotics which lead to Fokker-Planck equations [30]. In order to describe the asymptotic process, let us discuss in some details the evolution equation for the mean conviction, given by (2.6). For simplicity, and without loss of generality, let us assume λ and λ_B constant. Given a small parameter ϵ, the scaling

$$\lambda\to\epsilon\lambda, \quad \lambda_B\to\epsilon\lambda_B, \quad \kappa\to\sqrt{\epsilon}\kappa \tag{2.8}$$

is such that the mean value $M_C(t)$ satisfies

$$\frac{dM_C(t)}{dt} = -\epsilon\left(\lambda M_C(t) - \lambda_B M\right).$$

If we set $\tau=\epsilon t$, $F_\epsilon(x,\tau)=F(x,t)$, then

$$M_C(\tau) = \int_{\mathbb{R}_+} xF_\epsilon(x,\tau)\,dx = \int_{\mathbb{R}_+} xF(x,\tau)\,dx = M_C(t),$$

and the mean value of the density $F_\epsilon(x,\tau)$ solves

$$\frac{dM_C(\tau)}{d\tau} = -\lambda M_C(\tau) + \lambda_B M. \tag{2.9}$$

Note that equation (2.9) does not depend explicitly on the scaling parameter ϵ. In other words, we can reduce in each interaction the variation of conviction, waiting enough time to get the same law for the mean value of the knowledge density.

We can consequently investigate the situation in which most of the interactions produce a very small variation of conviction ($\epsilon\to 0$), while at the same time the evolution of the conviction density is such that (2.9) remains unchanged. We will call this limit quasi-invariant conviction limit.

Let now assume that the centered random variable κ has bounded moments at least of order $n=3$, with $\langle\kappa^2\rangle=\mu$. Then, proceeding as in [30] (cf. also Chapter 1 in [17]), we obtain that the density $F_\epsilon(x,\tau)$ solves the equation

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}_+} F_\epsilon(x,t)\varphi(x)\,dx = \\ &-\int_{\mathbb{R}_+} (\lambda(x)x - \lambda_B(x)M) + F_\epsilon(x,t)\varphi'(x)\,dx \\ &+\frac{\mu}{2}\int_{\mathbb{R}_+} H^2(x)F_\epsilon(x,t)\varphi''(x)\,dx + R_\epsilon(x,\tau),\end{aligned} \tag{2.10}$$

where the remainder R_ϵ is vanishing as $\epsilon\to 0$. Consequently, it is shown that, as $\epsilon\to 0$, the density $F_\epsilon(x,\tau)$ converges towards the density $G(x,\tau)$ solution of the Fokker-Planck equation

$$\begin{aligned}\frac{\partial G(x,\tau)}{\partial\tau} &= \frac{\mu}{2}\frac{\partial^2}{\partial x}\left(H(x)^2 G(x,\tau)\right) + \\ \frac{\partial}{\partial x}&\left((\lambda(x)x - \lambda_B(x)M)G(x,\tau)\right).\end{aligned} \tag{2.11}$$

The case in which $\lambda(x)=\lambda$ and $\lambda_B(x)=\lambda_B$ allows to get the explicit form of the steady distribution of conviction [17]. We will present two realizations of the asymptotic profile, that enlighten the consequences of the choice of a particular function $H(\cdot)$. First, let us consider the case in which $H(x)=x$. In this case, the Fokker–Planck equation (2.11) coincides with the one obtained in [30], related to the steady distribution of wealth in a multi–agent market economy. One obtains

$$G_\infty(x) = \frac{G_0}{x^{2+2\lambda/\mu}}\exp\left\{-\frac{2\lambda_B M}{\mu x}\right\}. \tag{2.12}$$

In (2.12) the constant G_0 is chosen to fix the total mass of $G_\infty(x)$ equal to one. Note that the steady profile is heavy

tailed, and the size of the polynomial tails is related to both λ and σ. Hence, the percentage of individuals with high conviction is decreasing as soon as the parameter λ of insecurity is increasing, and/or the parameter of self-thinking is decreasing. It is moreover interesting to note that the size of the parameter λ_B is important only in the first part of the x-axis, and contributes to determine the size of the number of undecided. Like in the case of wealth distribution, this solution has a large *middle class*, namely a large part of the population with a certain degree of conviction, and a small *poor class*, namely a small part of undecided people.

The second case refers to the choice $H(x) = \sqrt{x}$. Now, people with high conviction is more resistant to change (randomly) with respect to the previous case. On the other hand, if the conviction is small, $x < 1$, the individual is less resistant to change. Direct computations now show that the steady profile is given by

$$H_\infty(x) = H_0\, x^{-1+(2\lambda_B M)/\mu} \exp\left\{ -\frac{2\lambda}{\mu} x \right\}, \qquad (2.13)$$

where the constant H_0 is chosen to fix the total mass of $H_\infty(x)$ equal to one. At difference with the previous case, the distribution decays exponentially to infinity, thus describing a population in which there are very few agents with a large conviction. Moreover, this distribution describes a population with a huge number of undecided agents. Note that, since the exponent of x in $H_\infty(\cdot)$ is strictly bigger than -1, $H_\infty(\cdot)$ is integrable for any choice of the relevant parameters.

Other choices of the exponent ν in the range $0 < \nu \le 1$ do not lead to essential differences. The previous examples show that, despite the simplicity of the kinetic interaction (2.1), by acting on the coefficient of the random part κ one can obtain very different types of steady conviction distributions.

III. THE BOLTZMANN EQUATION FOR OPINION AND CONVICTION

In this section, we will join the kinetic model for conviction with the kinetic model for opinion formation introduced in [15]. This model belongs to a class of models in which agents are indistinguishable. In most of these models [17] an agent's *state* at any instant of time $t \ge 0$ is completely characterized by his opinion v. In agreement with the usual assumptions of the pertinent literature, the variable v varies continuously from -1 to 1, where -1 and 1 denote two (extreme) opposite opinions. A remarkable consequence of introducing a continuous distribution of opinions in the interval $[-1, 1]$ is that the extremal opinions do not play any particular rule.

The unknown in this model is the density (or distribution function) $f = f(v, t)$, where $v \in \mathfrak{I} = [-1, 1]$ and the time $t \ge 0$, whose time evolution is described, as shown later, by a kinetic equation of Boltzmann type.

The precise meaning of the density f is the following. Given the population to study, if the opinions are defined on a sub-domain $D \subset \mathfrak{I}$, the integral

$$\int_D f(v, t)\, dv$$

represents the *number* of individuals with opinion included in D at time $t > 0$. It is assumed that the density function is normalized to 1, that is

$$\int_{\mathfrak{I}} f(v, t)\, dv = 1.$$

As always happens when dealing with a kinetic problem in which the variable belongs to a bounded domain, this choice introduces supplementary mathematical difficulties in the correct definition of binary interactions. In fact, it is essential to consider only interactions that do not produce opinions outside the allowed interval, which corresponds to imposing that the extreme opinions cannot be crossed. This crucial limitation emphasizes the difference between the present *social* interactions, where not all outcomes are permitted, and the classical interactions between molecules, or, more generally, the wealth trades (cf. [17], Chapter 5), where the only limitation for trades was to insure that the post-collision wealths had to be non-negative.

In order to build a possibly realistic model, this severe limitation has to be coupled with a reasonable physical interpretation of the process of opinion forming. In other words, the impossibility of crossing the boundaries has to be a by-product of good modelling of binary interactions.

From a microscopic viewpoint, the binary interaction in [15] are described by the rules

$$\begin{aligned} v^* &= v - \gamma P(|v|)(v - w) + \Theta D(|v|), \\ w^* &= w - \gamma P(|w|)(w - v) + \tilde{\Theta} D(|w|). \end{aligned} \qquad (3.1)$$

In (3.1), the pair (v, w), with $v, w \in \mathfrak{I}$, denotes the opinions of two arbitrary individuals before the interaction, and (v^*, w^*) their opinions after exchanging information between each other and with the exterior. The coefficient $\gamma \in (0, 1/2)$ is a given constant, while Θ and $\tilde{\Theta}$ are random variables with the same distribution, with zero mean and variance σ^2, taking values on a set $\mathcal{B} \subseteq \mathbb{R}$. The constant γ and the variance σ^2 measure respectively the compromise propensity and the degree of spreading of opinion due to diffusion, which describes possible changes of opinion due to personal access to information (self-thinking). Finally, the functions $P(\cdot)$ and $D(\cdot)$ take into account the local relevance of compromise and diffusion for a given opinion.

Let us describe in detail the interaction on the right-hand side of (3.1). The first part is related to the compromise propensity of the agents, and the last contains the diffusion effects due to individual deviations from the average behavior. The presence of both the functions

$P(\cdot)$ and $D(\cdot)$ is linked to the hypothesis that openness to change of opinion is linked to the opinion itself, and decreases as one gets closer to extremal opinions. This corresponds to the natural idea that extreme opinions are more difficult to change. Various realizations of these functions can be found in [15]. In all cases, however, we assume that both $P(|v|)$ and $D(|v|)$ are non-increasing with respect to $|v|$, and in addition $0 \le P(|v|) \le 1$, $0 \le D(|v|) \le 1$. Typical examples are given by $P(|v|) = 1 - |v|$ and $D(|v|) = \sqrt{1 - v^2}$.

In the absence of the diffusion contribution ($\Theta, \tilde{\Theta} \equiv 0$), (3.1) implies

$$\begin{aligned} v^* + w^* &= v + w + \gamma(v - w)\left(P(|v|) - P(|w|)\right), \\ v^* - w^* &= \left(1 - \gamma(P(|v|) + P(|w|))\right)(v - w). \end{aligned} \tag{3.2}$$

Thus, unless the function $P(\cdot)$ is assumed constant, $P = 1$, the total *momentum* is not conserved and it can increase or decrease depending on the opinions before the interaction. If $P(\cdot)$ is assumed constant, the conservation law is reminiscent of analogous conservations which take place in kinetic theory. In such a situation, thanks to the upper bound on the coefficient γ, equations (3.1) correspond to a granular-gas-like interaction [17] where the stationary state is a Dirac delta centered on the average opinion. This behavior is a consequence of the fact that, in a single interaction, the compromise propensity implies that the difference of opinion is diminishing, with $|v^* - w^*| = (1 - 2\gamma)|v - w|$. Thus, all agents in the society will end up with exactly the same opinion. Note that in this elementary case a constant part of the relative opinion is restituted after the interaction. In all cases, however, the second inequality in (3.2) implies that the difference of opinion is diminishing after the interaction.

We remark, moreover, that, in the absence of diffusion, the lateral bounds are not violated, since

$$\begin{aligned} v^* = &\quad (1 - \gamma P(|v|))v + \gamma P(|v|)w, \\ w^* = &\quad (1 - \gamma P(|w|))w + \gamma P(|w|)v, \end{aligned} \tag{3.3}$$

imply

$$\max\{|v^*|, |w^*|\} \le \max\{|v|, |w|\}.$$

Let $f(v,t)$ denote the distribution of opinion $v \in \mathfrak{I}$ at time $t \ge 0$. The time evolution of f is recovered as a balance between bilinear gain and loss of opinion terms, described in weak form by the integro-differential equation of Boltzmann type

$$\begin{aligned} &\frac{d}{dt}\int_{\mathfrak{I}} \varphi(v) f(v,t)\,dv = (Q(f,f),\varphi) = \\ &\left\langle \int_{\mathfrak{I}^2} f(v)f(w)(\varphi(v^*) + \varphi(w^*) - \varphi(v) - \varphi(w))dvdw \right\rangle, \end{aligned} \tag{3.4}$$

where (v^*, w^*) are the post-interaction opinions generated by the pair (v, w) in (3.1).

We remark that equation (3.4) is consistent with the fact that a suitable choice of the function $D(\cdot)$ in (3.1) coupled with a small support $\mathcal{B}$ of the random variables implies that both $|v^*| \le 1$ and $|w^*| \le 1$.

The analysis in [15] essentially shows that the microscopic interaction (3.1) is so general that the kinetic equation (3.4) can describe a variety of different behaviors of opinion. In its original formulation, both the compromise and the self-thinking intensities were assigned in terms of the universal constant γ and of the universal random parameters $\Theta, \tilde{\Theta}$. Suppose now that these quantities in (3.1) could depend of the personal conviction of the agent. For example, one reasonable assumption would be that an individual with high personal conviction is more resistant to move towards opinion of any other agent by compromise. Also, an high conviction could imply a reduction of the personal self-thinking. If one agrees with these assumptions, the binary trade (3.1) has to be modified to include the effect of conviction. Given two agents A and B characterized by the pair (x, v) (respectively (y, w)) of conviction and opinion, the new binary trade between A and B now reads

$$\begin{aligned} v^* &= v - \gamma\,\Psi(x)P(|v|)(v - w) + \Phi(x)\Theta D(|v|), \\ w^* &= w - \gamma\,\Psi(y)P(|w|)(w - v) + \Phi(y)\tilde{\Theta} D(|w|). \end{aligned} \tag{3.5}$$

In (3.5) the personal compromise propensity and self-thinking of the agents are modified by means of the functions $\Psi = \Psi(x)$ and $\Phi = \Phi(x)$, which depend on the convictions parameters. In this way, the outcome of the interaction results from a combined effect of (personal) compromise propensity, conviction and opinion. Among other possibilities, one reasonable choice is to fix the functions $\Psi(\cdot)$ and $\Phi(\cdot)$ as non-increasing functions. This reflects the idea that the conviction acts to increase the tendency to remain of the same opinion. Among others, a possible choice is

$$\Psi(x) = (1 + (x - A)_+)^{-\alpha}, \quad \Phi(x) = (1 + (x - B)_+)^{-\beta}.$$

Here A, B, α, β are nonnegative constants, and $h(x)_+$ denotes the positive part of $h(x)$. By choosing $A > 0$ (respectively $B > 0$), conviction will start to influence the change of opinion only when $x > A$ (respectively $x > B$). It is interesting to remark that the presence of the conviction parameter (through the functions Ψ and Φ), is such that the post-interaction opinion of an agent with high conviction remains close to the pre-interaction opinion. This induces a mechanism in which the opinions of agents with low conviction are attracted towards opinions of agents with high conviction.

Assuming the binary trade (3.5) as the microscopic binary exchange of conviction and opinion in the system of agents, the joint evolution of these quantities is described in terms of the density $f = f(x, v, t)$ of agents which at time $t > 0$ are represented by their conviction $x \in \mathbb{R}_+$ and wealth $v \in \mathfrak{I}$. The evolution in time of the density f is described by the following kinetic equation (in weak

form) [17]

$$\frac{d}{dt}\int_{\mathbb{R}_+\times\mathfrak{I}} \varphi(x,v) f(x,v,t)\,dx\,dv =$$
$$\frac{1}{2}\Big\langle \int_{\mathbb{R}_+^2\times\mathfrak{I}^2} \big(\varphi(x^*,v^*)+\varphi(y^*,w^*)-\varphi(x,v)-\varphi(y,w)\big)\times$$
$$\times\, f(x,v,t) f(y,w,t) C(z)\,dx\,dy\,dz\,dv\,dw \Big\rangle. \tag{3.6}$$

In (3.6) the pairs (x^*,v^*) and (y^*,w^*) are obtained from the pairs (x,v) and (y,w) by (2.1) and (3.5). Note that, by choosing φ independent of v, that is $\varphi=\varphi(x)$, equation (3.6) reduces to the equation (2.3) for the marginal density of conviction $F(x,t)$.

To obtain analytic solutions to the Boltzmann-like equation (3.6) is prohibitive. The main reason is that the unknown density in the kinetic equation depends on two variables with different laws of interaction. In addition, while the interaction for conviction does not depend on the opinion variable, the law of interaction for the opinion does depend on the conviction. Also, at difference with the one-dimensional models, passage to Fokker-Planck equations (cf. [26] and the references therein) does not help in a substantial way. For this reason, we will resort to numerical investigation of (3.6), to understand the effects of the introduction of the conviction variable in the distribution of opinions.

IV. NUMERICAL EXPERIMENTS

This section contains a numerical description of the solutions to the Boltzmann-type equation (3.6). For the numerical approximation of the Boltzmann equation we apply a Monte Carlo method, as described in Chapter 4 of [17]. If not otherwise stated the kinetic simulation has been performed with $N=10^4$ particles.

The numerical experiments will help to clarify the role of conviction in the final distribution of the opinion density among the agents. The numerical simulations enhance the fact that the density $f(x,v,t)$ will rapidly converge towards a stationary distribution [17]. As usual in kinetic theory, this stationary solution will be reached in an exponentially fast time.

The numerical experiments will report the joint density of conviction and opinion in the agent system. The opinion variable will be reported on the horizontal axis, while the conviction variable will be reported on the vertical one. The color intensity will refer to the concentration of opinions. The following numerical tests have been considered.

Test 1

In the first test we consider the case of a conviction interaction where the diffusion coefficient in (2.1) is linear, $H(x)=x$. As described in Section II the distribution of conviction in this case is heavy tailed, with an important presence of agents with high conviction, and a large part of the population with a mean degree of conviction. In (3.5) we shall consider

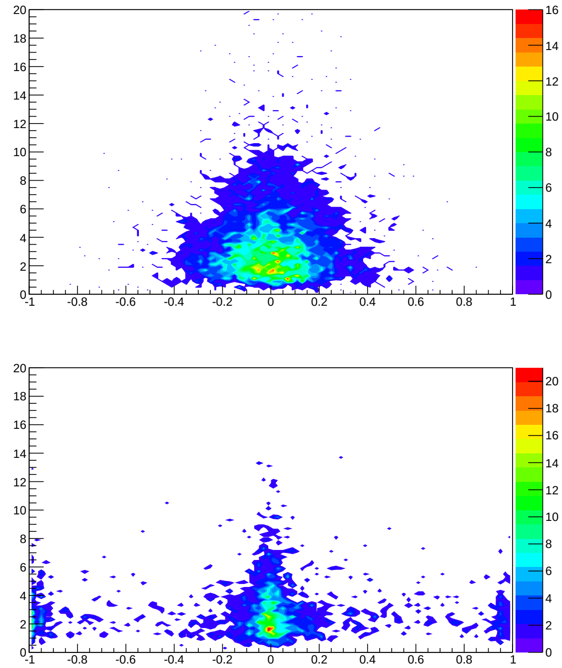

FIG. 4.1: Test 1: The particles solution with $N=10000$ particles and linear H. High diffusion in conviction and reduced self-thinking (up) compared to low diffusion in conviction and high self-thinking (down)

$$\Phi(x)=\Psi(x)=\frac{1}{1+x}.$$

We further take $\lambda=\lambda_B=0.5$ in (2.1), and $P(|v|)=1$, $D(|v|)=\sqrt{1-v^2}$ in (3.5). We consider a population of agents with an initially uniformly distributed opinion and a conviction uniformly distributed on the interval $[0,5]$. We choose a time step of $\Delta t=1$ and a final computation time of $t=50$, where the steady state is practically reached.

Since the evolution of the conviction in the model is independent from the opinion, the latter is scaled in order to fix the mean equal to 0. We report the results for the particle density corresponding to different values of μ, γ and the variance σ^2 of the random variables Θ and $\tilde{\Theta}$ in Figure 4.1. This allows to verify the essential role of the diffusion processes in conviction and opinion formation. In Figure 4.3 we plot the marginal densities together with the tail distribution

$$\bar{\mathcal{F}}(x)=1-\mathcal{F}, \qquad \bar{\mathcal{G}}(x)=1-\mathcal{G},$$

which are plotted in *loglog* scale to visualize the tails behavior.

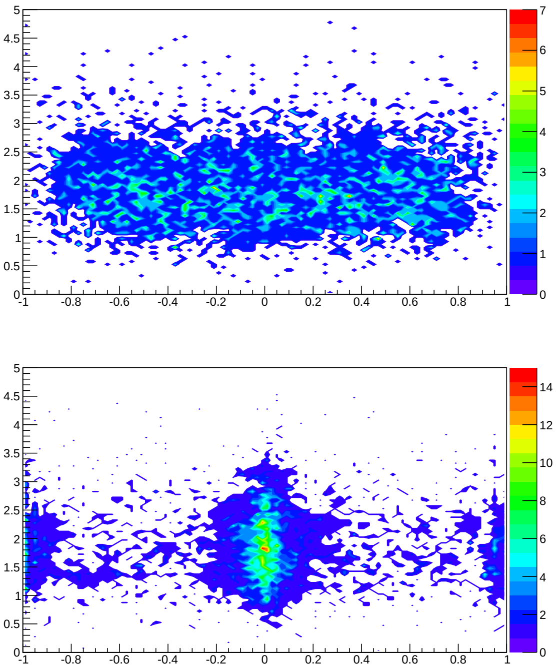

FIG. 4.2: Test 2: The particles solution with $N = 10000$ particles and $H(x) = \sqrt{x}$. High diffusion in conviction and reduced self-thinking (up) compared to low diffusion in conviction and high self-thinking (down).

Test 2

In this new test, we maintain the same values for the parameters, and we modify the diffusion coefficient in (2.1), which is now assumed as $H(x) = \sqrt{x}$. Within this choice, with respect to the previous test we expect the formation of a larger class on undecided agents. The results are reported in Figure 4.2 for the full density. At difference with the results of Test 1, opinion is spread out almost uniformly among people with low conviction. It is remarkable that in this second test, as expected, conviction is essentially distributed in the interval $[, 5]$, at difference with Test 1, where agents reach a conviction parameter of 20.

The same effect is evident in Figure 4.3, which refers to both Tests 1 and 2 in which, to understand the evolution in case of asymmetry, the initial distribution of opinions was chosen uniformly distributed on the positive part of the interval.

V. CONCLUSIONS

Opinion formation in a society of agents depends on many aspects, even if it appears to have very stable features, like formation of clusters. In this note, we introduced and discussed a kinetic model for the joint evolution of opinion in presence of conviction, based on the assumption that conviction is a relevant parameter that can influence the distribution of opinion by acting on the personal attitude to compromise, as well as in limiting the self-thinking. Numerical experiments put in evidence that the role of conviction relies in concentrating the final distribution of opinions towards a well-defined one.

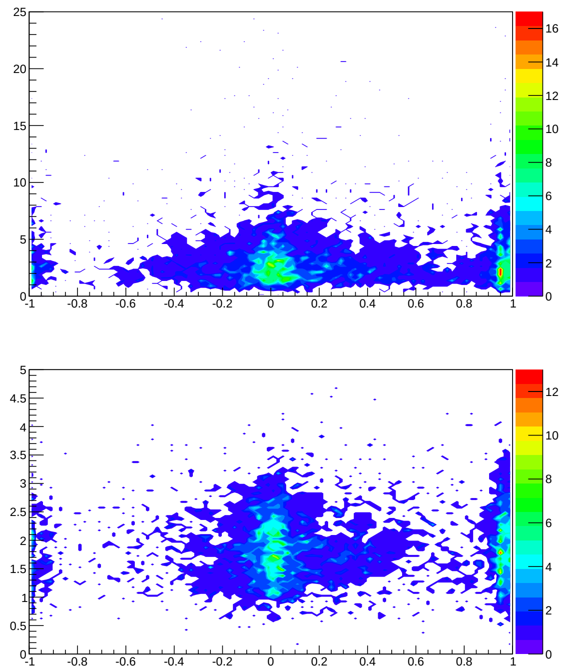

FIG. 4.3: Initial asymmetry in opinion leads to different opinion-conviction distributions. $H(x) = x$ (up), and $H(x) = \sqrt{x}$ (down).

Acknowledgement. This work has been done under the activities of the National Group of Mathematical Physics (GNFM). The support of the MIUR Research Projects of National Interest *"Optimal mass transportation, geometrical and functional inequalities with applications"* is kindly acknowledged. One of the authors (G.T.), thanks Eitan Tadmor for the interesting discussion about the role of conviction in opinion formation, which was the starting point for this research.

[1] E. Ben-Naim, P. L. Krapivski and S. Redner, *Physica D* **183**, 190 (2003).

[2] E. Ben-Naim, P. L. Krapivski, R. Vazquez and S. Redner, *Physica A* **330**, 99 (2003).

[3] E. Ben-Naim, *Europhys. Lett.* **69**, 671 (2005).

[4] G. Deffuant, F. Amblard, G. Weisbuch, and T. Faure, *J. Art. Soc. Soc. Sim.* **5**, 1 (2002)

[5] S. Galam, Y. Gefen, and Y. Shapir, *J. Math. Sociology* **9**, 1 (1982).

[6] S. Galam, and J.D. Zucker, *Physica A* **287**, 644 (2000).

[7] K. Sznajd–Weron, and J. Sznajd, *Int. J. Mod. Phys. C* **11**, 1157 (2000).

[8] C. Castellano, S. Fortunato, and V. Loreto, *Rev. Mod. Phys.* **81** 591 (2009).

[9] A. Chatterjee, e-print arXiv:1409.8030. To be published in *Econophysics and Data Driven Modelling of Market Dynamics*, Eds. F Abergel, H. Aoyama, B.K. Chakrabarti, A. Chakraborti, A. Ghosh, New Economic Windows, Springer (2015).

[10] M.L. Bertotti, and M. Delitala, *Math. Comp. Model.* **48**, 1107 (2008).

[11] L. Boudin, and F. Salvarani, *Kinetic Rel. Mod.* **2**, 433 (2009).

[12] L. Boudin, and F. Salvarani, *ESAIM: Math. Mod. Num. Anal.* **43**, 507 (2009).

[13] L. Boudin, A. Mercier, and F. Salvarani, *Physica A* **391**, 5672 (2012).

[14] V. Comincioli, L. Della Croce, and G. Toscani, *Kinetic Rel. Mod.* **2**, 135 (2009).

[15] G. Toscani, *Commun. Math. Sci.* **4**, 481 (2006).

[16] S. Motsch, and E. Tadmor, *SIAM Rev.* **56**, 577 (2014).

[17] L. Pareschi, and G. Toscani, *Interacting multiagent systems. Kinetic equations & Monte Carlo methods*, Oxford University Press, Oxford (2013)

[18] W. Weidlich, *Sociodynamics: a Systematic Approach to Mathematical Modelling in the Social Sciences.* Harwood Academic Publishers, Amsterdam (2000).

[19] B. Düring, M. Markowich, J.F. Pietschmann, and M.T. Wolfram, *Proc. R. Soc. A* **465**, 3687 (2009)

[20] M. Lallouache, A. Chakrabarti, A. Chakraborti, and B. Chakrabarti, *Phys. Rew. E* **82**, 056112 (2010).

[21] A. Chakraborti, and B.K. Chakrabarti, *Eur. Phys. J. B* **17**, 167 (2000)

[22] P. Sen, *Phys. Rev. E* **83**, 016108 (2011).

[23] S. Biswas, *Phys. Rev, E* **84**, 056106 (2011).

[24] N. Crokidakis, *J. Stat. Mech.* (7), P07008 (2013).

[25] N. Crokidakis, and C. Anteneodo, *Phys. Rev. E* **86**, 061127 (2012).

[26] L. Pareschi, and G. Toscani, *Phil. Trans. R. Soc. A* **372**, 20130396, 6 October (2014).

[27] R. Hegselman and U. Krause, *JASSS J. Art. Soc. Soc. Sim.* **5**, 3 (2002)

[28] G.Naldi, L.Pareschi, and G.Toscani Eds. *Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences*, Birkhauser, Boston (2010).

[29] R.C. Teevan, and R.C. Birney, *Readings for Introductory Psychology* Harcourt, Brace & World, New York, (1965).

[30] S. Cordier, L. Pareschi, and G. Toscani, *J. Stat. Phys.* **120**, 253 (2005).